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Lamb shift and Stark effect in simultaneous space-space and momentum-momentum noncommutative quantum mechanics and θ-deformed $su(2)$ algebra.

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We study the spectrum of Hydrogen atom, Lamb shift and Stark effect in the framework of simultaneous space-space and momentum-momentum (s-s , p-p) noncommutative quantum mechanics. The results show that the widths of Lamb shift due to noncommutativity is bigger than the one presented in [1]. We also study the algebras of observables of systems of identical particles in s-s , p-p noncommutative quantum mechanics. We intoduce θ-deformed $su(2)$ algebra.

Introduction.

It is generally believed that the picture of space-time as a manifold should break down at very short distances of the order of the Planck length. Field theories on noncommutative spaces may play an important role in unraveling the properties of nature at the Planck scale. The study on noncommutative spaces is much important for understanding phenomena at short distances beyond the present test of QED. It has been shown that the noncommutative geometry naturally appears in string theory with a non zero antisymmetric B-field [8]. Besides the string theory arguments the noncommutative field theories by themselves are very interesting. In recent years there have been a lot of work devoted to the study of NCFT's(or NCQM) and possible experimental consequences of extensions of the standard formalism to noncommutative one (see e.g.[1-25]).
In field theories the noncommutativity is introduced by replacing the standard product by the star product. NCQM is formulated in the same way as the standard quantum mechanics SQM (quantum mechanics in commutative spaces), that is in terms of the same dynamical variables represented by operators in a Hilbert space and a state vector that evolves according to the Schroedinger equation :

$$i\frac{d}{dt}|\psi>= H_{nc}|\psi>, \tag{1}$$

we have taken into account $\hbar = 1$. $H_{nc} \equiv H_\theta$ denotes the Hamiltonian for a given system in the noncommutative space. In the literatures two approaches

have been considered for constructing the NCQM :
a) $H_\theta = H$, so that the only difference between SQM and NCQM is the presence of a nonzero θ in the commutator of the position operators.
b) By deriving the Hamiltonian from the Moyal analog of the standard Schroedinger equation :

$$i\frac{\partial}{\partial t}\psi(x,t) = H(p = \frac{1}{i}\nabla, x) * \psi(x,t) \equiv H_\theta \psi(x,t), \tag{2}$$

where $H(p,x)$ is the same Hamiltonian as in the standard theory, and as we observe the θ - dependence enters now through the star product [5]. In [6], it is shown that these two approaches lead to the same physical theory.
In order to specify the phase space and the Hilbert space on which operators act one can take the Hilbert space to be exactly the same as the Hilbert space of Corresponding commutative systems [1]. There are different types of noncommutative theories [9]. For the phase space we consider both space-space and momentum -momentum noncommutativity. The space-space noncommutativity is inferred from the string theory [7,8]. The motivation for considering momentum-momentum noncommutativity are as follows :
a). To incorporate an additional background magnetic field [9,10].
b). To maintain Bose-Einstien statistics for systems of identical Bosons is constructed by generalizing one-particle quantum mechanics [12].
The noncommutative space can be realized by the coordinate operators satisfying :

$$[\hat{x}_i, \hat{x}_j] = i\zeta^{-2}\Lambda_{NC}^{-2} d\theta_{ij} \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \tag{3}$$

where $\theta_{ij} = \epsilon_{ijk}\theta_k$, is the noncommutativity parameter. Λ_{NC} is the NC energy scale, and d is a constant frame-independent dimensionless parameter. The scaling factor ζ will be defined later. In this paper we put $\theta_3 = \theta$ and the rest of the θ-components to zero , which can be done by a rotation or a redifinition of coordinates. The NC coordinates $\hat{x}$ and momentum $\hat{p}$ in equ.(3), can be expressed in terms of commutative coordinates x and p as follows :

$$\hat{x}_i = x_i - \frac{1}{2}\zeta^{-2}\Lambda_{NC}^{-2} d\theta_{ij} p_j, \qquad \hat{p}_i = p_i \tag{4}$$

where now x and p satisfy in usual canonical commutation relations :

$$[x_i, x_j] = [p_i, p_j] = 0, \qquad [x_i, p_j] = i\delta_{ij}, \tag{5}$$

In the case of simultaneous space-space and momentum-momentum noncommutativity, the consistent NCQM algebra is :

$$[\hat{x}_i, \hat{x}_j] = i\zeta^{-2}\Lambda_{NC}^{-2} d\theta_{ij} \qquad [\hat{p}_i, \hat{p}_j] = i\zeta^{-2}\Lambda_{NC}^{2} d'\theta_{ij} \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \tag{6}$$

d' is another constant frame-independent dimensionless parameter. The scaling factor ζ is defined as :

$$\zeta = (1 + \frac{dd'}{4})^{\frac{1}{2}} \tag{7}$$

The NC coordinates $\hat{x}$ and momentum $\hat{p}$ can be written in terms of usual coordinates x and momentum p :

$$\hat{x}_i = x_i - \frac{1}{2}\zeta^{-2}\Lambda_{NC}^{-2} d\theta_{ij} p_j \qquad \hat{p}_i = p_i + \frac{1}{2}\zeta^{-2}\Lambda_{NC}^{2} d'\theta_{ij} x_j \tag{8}$$

where x and p satisfy in equ.(5). We know that all the two dimensional antisymmetric tensors can be represented by the unit two dimensional antisymmetric tensor ϵ_{ij}. The difference of the tensorial forms of the $x-x$ and $p-p$ commutators are represented by different cofficients d and d'. The dimensional parameters Λ_{NC}^{-2} and Λ_{NC}^{2} guarantee the correct dimensional of the tensorial forms of $x-x$ and $p-p$ commutators.

2. algebras of observables of systems of identical particles in s-s, p-p noncommutative two dimensional spaces. θ-deformed algebras.

Heisenberg quantization for systems of identical particles in commutative case has been studied in detail in [26]. In this section we apply the Heisenberg quantization to systems of two identical particles in s-s,p-p noncommutative two dimensional spaces. The three dimensional case can be analyzed along similar lines. We can describe the two particles systems by relative coordinates in the same way as in one dimensional case. We define complex quantities $\hat{a}_{j\pm}$ as follows :

$$\hat{a}_{j\pm} = \sqrt{\frac{\mu\omega}{2}}(\hat{x}_j \pm \frac{i}{\mu\omega}\hat{p}_j), \qquad j = 1, 2. \tag{9}$$

where $\hat{x}_\ell$ and $\hat{p}_\ell$ satisfy in equ.(6).

It is shown in [12] that to maintain Bose-Einstein statistics the basic assumption is that operators $\hat{a}_i^\dagger$ and $\hat{a}_j^\dagger$ are commuting. This requirement leads to a consistency condition $d' = \mu^2\omega^2\Lambda_{NC}^{-4} d$.Then we have:

$$[\hat{a}_{j+}, \hat{a}_{k+}] = [\hat{a}_{j-}, \hat{a}_{k-}] = 0 \tag{10}$$

$$[\hat{a}_{j-}, \hat{a}_{k+}] = \delta_{jk} + i\beta\theta_{jk}. \tag{11}$$

where $\beta = \zeta^{-2}\mu\omega\Lambda_{NC}^{-2} d$. The generalized one-dimensional observables A_j, B_j and C_j are :

$$\hat{A}_j = \frac{1}{4}(\hat{a}_{j+}\hat{a}_{j-} + \hat{a}_{j-}\hat{a}_{j+}) \qquad \hat{B}_{j\pm} = \hat{B}_j \pm i\hat{C}_j = \frac{1}{2}(\hat{a}_{j\pm})^2. \tag{12}$$

In addition we have two-dimensional observables which are the real and imaginary parts of :

$$\hat{D}_\pm = \hat{D}_{re} \pm i\hat{D}_{im} = a_{1\pm}a_{2\pm} \qquad \hat{E}_\pm = \hat{E}_{re} \pm i\hat{E}_{im} = \hat{a}_{1\mp}\hat{a}_{2\pm}. \tag{13}$$

There are two $sp(1,R)$ algebras $\hat{A}_1$, $\hat{B}_{1\pm}$ and $\hat{A}_2$, $\hat{B}_{2\pm}$:

$$[\hat{A}_j, \hat{B}_{j\pm}] = \pm\hat{B}_{j\pm} \qquad [\hat{B}_{j-}, \hat{B}_{j+}] = 2\hat{A}_j \quad j = 1, 2. \tag{14}$$

There are also two other algebras, one θ-deformed $sp(1,R)$ algebra :

$$[\hat{A}_1 + \hat{A}_2, D_\pm] = \pm \hat{D}_\pm, \tag{15}$$

$$[\hat{D}_+, \hat{D}_-] = -2(\hat{A}_1 + \hat{A}_2) + \theta\beta L, \tag{16}$$

where L is the relative angular momentum operator.
and one θ-deformed $su(2)$ algebra :

$$[\hat{A}_2 - \hat{A}_1, \hat{E}_\pm] = \pm \hat{E}_\pm - \frac{i}{2}\theta\beta[2(\hat{A}_1 + \hat{A}_2) + 1] \tag{17}$$

$$[\hat{E}_+, \hat{E}_-] = 2(\hat{A}_2 - \hat{A}_1). \tag{18}$$

where N is the number operator. To make this deformed algebra more familiar we use Schwinger's model notation

$$J_+ = a_{2-}a_{1+} \qquad\qquad J_- = a_{1-}a_{2+} \tag{19}$$

$$J_z = \frac{1}{2}(a_{2-}a_{2+} - a_{1-}a_{1+}) \qquad N = (a_{2-}a_{2+} + a_{1-}a_{1+}) \tag{20}$$

one can easily show that :

$$A_2 - A_1 = \frac{1}{2}(a_{2-}a_{2+} - a_{1-}a_{1+}) = J_z \tag{21}$$

$$A_2 + A_1 = \frac{1}{2}(a_{2-}a_{2+} + a_{1-}a_{1+} - 1) = \frac{1}{2}(N-1) \tag{22}$$

Eqs.(17-22) lead us to the following θ-deformed $su(2)$ algebra :

$$[J_+, J_-] = 2J_z \tag{23}$$

$$[J_z, J_\pm] = \pm J_\pm - i\frac{1}{2}\theta\beta N \tag{24}$$

To study the representation of this algebra we introduce operators j_+ and j_- as follows :

$$j_+ = J_+ - \frac{i}{2}\theta\beta N. \tag{25}$$

$$j_- = J_- - \frac{i}{2}\theta\beta N, \qquad\qquad j_z = J_z \tag{26}$$

One can show that j_+, j_- and j_z satisfy the ordinary $su(2)$ algebra :

$$[j_+, j_-] = 2j_z \tag{27}$$

$$[j_z, j_\pm] = \pm j_\pm \tag{28}$$

Now everything goes as usual because representations of $su(2)$ algebra are well known.

The other commutation relations are as follows :

$$[\hat{E}_-, \hat{D}_+] = 2\hat{B}_{1+} - i\theta\beta D_+. \tag{29}$$

$$[\hat{D}_-, \hat{E}_+] = 2\hat{B}_{1-} + i\theta\beta D_-. \tag{30}$$

$$[\hat{E}_+, \hat{D}_+] = 2\hat{B}_{2+} + i\theta\beta D_+. \tag{31}$$

$$[\hat{D}_-, \hat{E}_-] = 2\hat{B}_{2-} - i\theta\beta D_-, \tag{32}$$

$$[\hat{E}_-, \hat{B}_{2+}] = \hat{D}_+. \tag{33}$$

$$[\hat{E}_+, \hat{B}_{1+}] = \hat{D}_+. \tag{34}$$

$$[\hat{E}_-, \hat{B}_{1-}] = -\hat{D}_-. \tag{35}$$

$$[\hat{E}_+, \hat{B}_{2-}] = -\hat{D}_-. \tag{36}$$

$$[\hat{D}_-, \hat{B}_{1+}] = \hat{E}_- - i\theta\beta a_{1-}a_{1+}, \tag{37}$$

$$[\hat{D}_+, \hat{B}_{2-}] = -\hat{E}_- + i\theta\beta a_{2-}a_{2+} \tag{38}$$

$$[\hat{D}_-, \hat{B}_{2+}] = \hat{E}_+ + i\theta\beta a_{2+}a_{2-}. \tag{39}$$

$$[\hat{D}_+, \hat{B}_{1-}] = -\hat{E}_+ - i\theta\beta a_{1+}a_{1-}. \tag{40}$$

3. Hydrogen atom spectrum ,Lamb shift and Stark effect in simultaneous s-s,p-p noncommutative spaces.

Now we study the spectrum of Hydrogen atom, Lamb shift and Stark effect in the case of simultaneous space-space and momentum-momentum noncommutativity and compare with the results presented in [1].

Using equ.(8) the kinetic and potential terms can be written as follows :

$$\frac{\vec{\hat{P}}}{2m} = \frac{\vec{P}}{2m} - \frac{\beta}{2m}\vec{L}.\vec{\theta} + O(\theta^2). \tag{41}$$

$$V(\hat{r}) = V(r) - \frac{Ze^2\beta}{4r^3}\vec{L}.\vec{\theta} + O(\theta^2). \tag{42}$$

using the fact that $L.\theta = L_z\theta$ and :

$$\langle\ \ell j j_z \mid L_z \mid \acute{\ell}\acute{j}\acute{j_z}\rangle = j_z\left(1 \mp \frac{1}{2\ell+1}\right)\delta_{\ell\acute{\ell}}\delta_{j_z\acute{j_z}}, \qquad j = \ell \pm \frac{1}{2} \tag{43}$$

the energy level shift by (40) and (41) become :

$$\Delta E_{NC}^{H-atom} = -\left[\frac{m_e}{4}(Z\alpha)^4\frac{\theta}{\lambda_e^2}\beta f_{n,\ell} + \frac{\theta\beta}{2m}\right] j_z\left(1 \mp \frac{1}{2\ell+1}\right)\delta_{\ell\acute{\ell}}\delta_{j_z\acute{j_z}} \tag{44}$$

As it is observed and mentioned in [1], Lamb shift i.e. $2P_{\frac{1}{2}} \rightarrow 2S_{\frac{1}{2}}$ transition differs from the usual commutative case in which the shift depends only on the ℓ quantum number and all the corrections are due to the field theory

loop effects. The Lamb shift for the simultaneous space-space and momentum-momentum noncommutative H-atom, besides the usual loop effects, depends on the j_z quantum number. There is also a new channel which is opened because of noncommutativity : $2P_{\frac{1}{2}}^{-\frac{1}{2}} \to 2P_{\frac{1}{2}}^{\frac{1}{2}}$. The usual Lamb shift , $2P_{\frac{1}{2}} \to 2S_{\frac{1}{2}}$ is now split into two parts, $2P_{\frac{1}{2}}^{-\frac{1}{2}} \to 2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}^{\frac{1}{2}} \to 2S_{\frac{1}{2}}$, which means that the simultaneous noncommutativity effects increase the widths and split the Lamb shift line by a factor proportional to θ, but the gap widths between two parts is bigger than the case of single space-space noncommutativity(i.e. the results presented in [1]). only experiments can tell whether the space is really non-commutative or not, and in case it is, which is the non-commutative structure, simultaneous space-space and momentum-momentum or only space-space non-commutativity.

Now let us study the Stark effect in the case of simultaneous s-s and p-p noncommutativity. The potential energy of the atomic electron in an external electric field oriented along the z axis is given by :

$$V_{Stark} = eEz + \frac{e\beta}{4}\,(\theta \times p)\cdot E + \frac{\beta}{2m}\,(\theta \times p)\cdot (r^3 E) \tag{45}$$

To the first order in perturbation theory the contribution to the Stark effect due to the second term is zero [1]. For the third term to the first order we have :

$$\Delta_{Stark}^{NC} \propto (\vec{\theta} \times \vec{E})_i \left\langle n\ell' j j'_z \left| p_i r^3 \right| n\ell j j_z \right\rangle \neq 0 \tag{46}$$

which is different from the result presented in [1].

Conclusion.

In conclusion we have studied the spectrum of Hydrogen atom , Lamb shift and Stark effect in simultaneous space-space and momentum-momentum noncommutative spaces which are different from those in the case of space-space noncommutativity. As we mentioned in the text only experiments can tell whether the space is really noncommutative or not, and in case it is, which is the noncommutative structure. We have also introduced new deformed $su(2)$ algebra which appears when one study Schwinger's model or quantization of systems of identical particles in simultaneous space-space and momentum-momentum noncommutative spaces.

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